The Economics of Early Warfare Over Land

Online Appendix: Proofs of Formal Propositions

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<u>Lemma 1</u> (x_A and x_B as functions of σ).

(a) From (8) $x_A(\sigma) + x_B(\sigma) = (s_A + s_B)/\phi(s_A, s_B)$, where $\phi(s_A, s_B) \equiv s_A/[1 + \sigma^{-1/(1-\alpha)}]^\alpha + s_B/[1 + \sigma^{1/(1-\alpha)}]^\alpha$ from (5b). Using $\sigma \equiv s_A/s_B > 0$ from (5a), some algebra shows that $x_A(\sigma) + x_B(\sigma) > 1$ iff the following expression is positive:

$$\sigma[(2 + \sigma^{-1/(1-\alpha)} + \sigma^{1/(1-\alpha)})^\alpha - (1 + \sigma^{1/(1-\alpha)})^\alpha]$$

$$+ (2 + \sigma^{-1/(1-\alpha)} + \sigma^{1/(1-\alpha)})^\alpha - (1 + \sigma^{-1/(1-\alpha)})^\alpha$$

This is true because each line is strictly positive.

(b) From (8) we have $x_A(\sigma) \equiv s_A/\phi(s_A, s_B)$. From (4) and (5c) this can be rewritten as

$x_A(\sigma) = s_A / \max \{s_A L_A^\alpha + s_B L_B^\alpha$ subject to $L_A \geq 0, L_B \geq 0, L_A + L_B = 1\}$ or

$x_A(\sigma) = 1 / \max \{L_A^\alpha + (1/\sigma)L_B^\alpha$ subject to $L_A \geq 0, L_B \geq 0, L_A + L_B = 1\}$

The right hand side is increasing in σ due to the envelope theorem. Thus $x_A'(\sigma) > 0$ for all $\sigma > 0$. A similar argument shows that $x_B'(\sigma) < 0$ for all $\sigma > 0$.

(c) Write $x_A(\sigma) = 1 / \max \{L_A^\alpha + (1/\sigma)L_B^\alpha$ subject to $L_A \geq 0, L_B \geq 0, L_A + L_B = 1\}$ as in (b). Because multiplication of the objective function by the constant $(1/s_A)$ has no effect on the solution, the optimal L_A and L_B in the denominator are given by (5d) for n = 1. Making this substitution, it can be shown that the denominator of

$x_A(\sigma)$ approaches infinity as $\sigma \to 0$ and approaches 1 as $\sigma \to \infty$. This gives $x_A(0) = 0$ and $x_A(\infty) = 1$.

(d) The proof parallels (c).

Proposition 1 (war and peace).

Attack is a dominant strategy for A iff $p_A h(n_A) > s_A n_A^{\alpha-1}$ or equivalently $n_A/N > x_A(\sigma)$. Attack is a dominant strategy for B iff $p_B h(n_B) > s_B n_B^{\alpha-1}$ or equivalently $n_B/N > x_B(\sigma)$. When neither inequality holds, there is peace as in (7). If both A and B attack, then $n_A/N + n_B/N > x_A(\sigma) + x_B(\sigma) > 1$ where the second inequality is obtained from (9a). This is impossible because $n_A + n_B = N$. Thus A and B cannot both attack. The results in (a), (b), and (c) follow from the first three sentences above.

To show that equality of marginal products implies peace, fix $N > 0$ and $\sigma > 0$. Write $L_A^* = N/[1 + \sigma^{-1/(1-\alpha)}]$ and $L_B^* = N/[1 + \sigma^{1/(1-\alpha)}]$ as in (5d), where $L_A^* + L_B^* = N$. This is the unique labor allocation that equates marginal products across sites, and it is also the unique allocation that equates average products across sites. Let the total food output from (L_A^*, L_B^*) be

$$Y^* = s_A(L_A^*)^{\alpha} + s_B(L_B^*)^{\alpha}$$
$$= H(N) = \max \{s_A L_A^{\alpha} + s_B L_B^{\alpha} \text{ subject to } L_A \geq 0, L_B \geq 0, L_A + L_B = N\}.$$

Because the average products are equal, we have $Y^*/N = s_A(L_A^*)^{\alpha-1} = s_B(L_B^*)^{\alpha-1}$. Peace is strictly better than war for group A when $Y^*/N = s_A(L_A^*)^{\alpha-1} > (L_A^*/N)h(L_A^*) = H(L_A^*)/N$. This holds because $L_A^* < N$ gives $H(L_A^*) < H(N) = Y^*$. The proof is similar for B. This shows that peace is strictly better for each group, so case (b) holds with strict inequalities.

Proposition 2 (interior locational equilibria).

For any given allocation n there are three possibilities: (a) B attacks; (b) there is peace; or (c) A attacks. Using Proposition 1, we consider each case in turn.

(a) B attacks iff $n_A/N < 1-x_B$ or equivalently $n_A/n_B < (1-x_B)/x_B$. The utility functions are those from the warfare case in (10). The inequalities in (11) yield $\eta^{1/\alpha} \le n_A/n_B \le (1/\eta)^{1/\alpha}$. Together this gives Proposition 2(a).

(b) There is peace iff $1-x_B \le n_A/N \le x_A$ or equivalently $(1-x_B)/x_B \le n_A/n_B \le x_A/(1-x_A)$. The utility functions are those from the peace case in (10). The inequalities in (11) yield $(\sigma\eta)^{1/(1-\alpha)} \le n_A/n_B \le (\sigma/\eta)^{1/(1-\alpha)}$. Together this gives Proposition 2(b).

(c) A attacks iff $x_A < n_A/N$ or equivalently $x_A/(1-x_A) < n_A/n_B$. The utility functions are those from the warfare case in (10). The inequalities in (11) yield $\eta^{1/\alpha} \le n_A/n_B \le (1/\eta)^{1/\alpha}$. Together this gives Proposition 2(c).

<u>Proposition 3</u> (migration).

(a) Follows from Definition 1, the construction of LE_B, LE_P, and LE_A in Proposition 2, and Definition 2(a).

(b) Suppose LE_B is empty and consider two possibilities:

(i) $m_A/m_B < (1-x_B)/x_B$, which yields war; and

(ii) $(1-x_B)/x_B \le m_A/m_B$ with m_A/m_B below LE_P, which yields peace.

In case (i), the utility functions for war in (10) give $u_A(n) = [\phi(s_A, s_B)/N]n_A^\alpha$ and $u_B(n) = [\phi(s_A, s_B)/N]n_B^\alpha$ for all n with $n_A \le m_A$. The fact that LE_B is empty implies $(1-x_B)/x_B \le \eta^{1/\alpha}$ so we have $n_A/n_B < \eta^{1/\alpha}$ for all such n. This gives $u_A(n) < \eta u_B(n)$ in (11) for all such n. From Definition 2(c) we obtain the final allocation n = (0, N).

In case (ii), the utility functions for peace in (10) give $u_A(n) = s_A n_A^{\alpha-1}$ and $u_B(n) = s_B n_B^{\alpha-1}$ for all n with $n_A \geq m_A$ and n_A/n_B below LE_P. From the construction of LE_P, this implies $\eta u_A(n) > u_B(n)$ for all such n. The allocation n with the smallest $n_A \geq m_A$ such that $\eta u_A(n) = u_B(n)$ is the one where n_A/n_B equals the lower bound of LE_P.

(c) Suppose LE_B is non-empty and consider two possibilities:

(i) m_A/m_B is below the lower bound of LE_B, which yields war;

(ii) m_A/m_B is between the upper bound of LE_B and the lower bound of LE_P, which may yield either war or peace.

In case (i), the argument is the same is in (b)(i) above, except that now $n_A/n_B < \eta^{1/\alpha}$ follows from the fact that all allocations under consideration have n_A/n_B below the lower bound of LE_B.

In case (ii), suppose $(1/\eta)^{1/\alpha} < m_A/m_B < (1-x_B)/x_B$, which yields war. The utility functions for war in (10) give $u_A(n) = [\phi(s_A, s_B)/N]n_A^{\alpha}$ and $u_B(n) = [\phi(s_A, s_B)/N]n_B^{\alpha}$ for all n with $n_A \geq m_A$ and $n_A/n_B < (1-x_B)/x_B$. From $(1/\eta)^{1/\alpha} < m_A/m_B \leq n_A/n_B$, at any such n we have $u_B(n) < \eta u_A(n)$. This includes the allocation m. From Definition 2(b), no such n can be a final allocation. Now consider n with $n_A/n_B \geq (1-x_B)/x_B$ where n_A/n_B is below the lower bound of LE_P. Any such n yields peace. At $n_A/n_B = (1-x_B)/x_B$ the function $u_B(n)$ is continuous while $u_A(n)$ has an upward jump. This maintains $u_B(n) < \eta u_A(n)$. From the construction of LE_P, the allocation n with the smallest $n_A \geq m_A$ such that $u_B(n) = \eta u_A(n)$ is the one where n_A/n_B equals the lower bound of LE_P. If instead the initial allocation has $m_A/m_B \geq (1-x_B)/x_B$ and m_A/m_B is below the lower bound of LE_P, we repeat the last part of the argument above.

(d) The argument is symmetric to case (b) above.

(e) The argument is symmetric to case (c) above.

Lemma 2 (group size ratio).

Fix $\sigma^t \in (0, \infty)$ and $m_A^t/m_B^t \in (0, \infty)$. We proceed in the following steps.

(a) Recall the definitions of $x_A(\sigma)$ and $x_B(\sigma)$ in (8). From Proposition 2, the pair (σ^t, m_A^t/m_B^t) determines whether m_A^t/m_B^t is in one of the sets LE_B, LE_P, or LE_A. If it is, then from Definition 2(a) we have $n_A^t/n_B^t = m_A^t/m_B^t$.

(b) If m_A^t/m_B^t is not in one of the sets LE_B, LE_P, or LE_A, this and the fact that m^t is interior implies that one of the four cases (b)-(e) in Proposition 3 applies. From Proposition 2, σ^t determines whether LE_B is empty or non-empty and likewise for LE_A. The ratios (σ^t, m_A^t/m_B^t) together determine which of (b)-(e) in Proposition 3 applies, and also the final allocation n^t, where n^t must be $(N^t, 0)$, $(0, N^t)$, the lower bound of LE_P, or the upper bound of LE_P.

(c) If step (a) applies with $m_A^t/m_B^t \in LE_P$, or step (b) applies and n^t is the lower or upper bound of LE_P, there is peace in period t. This follows because all ratios in the LE_P interval satisfy the conditions for peace in Proposition 1 by construction. We then obtain m_A^{t+1}/m_B^{t+1} from (12), where the new allocation m^{t+1} is interior.

(d) If step (a) applies with $m_A^t/m_B^t \in LE_B$ or $m_A^t/m_B^t \in LE_A$, there is a non-trivial war in period t and we obtain m_A^{t+1}/m_B^{t+1} from (13), where m^{t+1} is interior. If step (b) applies and n^t is $(N^t, 0)$ or $(0, N^t)$, there is a trivial war in period t, and again we obtain m_A^{t+1}/m_B^{t+1} from (13), where m^{t+1} is interior.

Proposition 4 (war and peace with Malthusian dynamics).

From Proposition 3, a non-trivial war occurs in period t+1 iff $m_A^t/m_B^t \in LE_B$ or $m_A^t/m_B^t \in LE_A$. In all other cases, either $n^{t+1} = (N^{t+1}, 0)$ or $n^{t+1} = (0, N^{t+1})$ so there is a trivial war; or $n_A^{t+1}/n_B^{t+1} \in LE_P$ so there is peace.

(a) Proposition 2(a) shows that a necessary condition for $m_A^{t+1}/m_B^{t+1} \in LE_B$ is $\eta^{1/\alpha} \leq m_A^{t+1}/m_B^{t+1} \leq (1/\eta)^{1/\alpha}$. Proposition 2(c) shows that the same condition is necessary for $m_A^{t+1}/m_B^{t+1} \in LE_A$.

(b) Proposition 2(a) shows that a necessary condition for $m_A^{t+1}/m_B^{t+1} \in LE_B$ is $m_A^{t+1}/m_B^{t+1} < [1-x_B(\sigma^{t+1})]/x_B(\sigma^{t+1})$. Proposition 2(c) shows that a necessary condition for $m_A^{t+1}/m_B^{t+1} \in LE_A$ is $m_A^{t+1}/m_B^{t+1} > x_A(\sigma^{t+1})/[1-x_A(\sigma^{t+1})]$.

When the necessary condition in (a) is combined with the necessary condition for LE_B in (b), by Proposition 2 this suffices for $m_A^{t+1}/m_B^{t+1} \in LE_B$. The result for LE_A is the same.

We want to show that one of the inequalities in (b) holds iff $\sigma^{t+1} \notin [\sigma_A^{t+1}, \sigma_B^{t+1}]$ as in Proposition 4(b). The solutions for σ_A^{t+1} and σ_B^{t+1} exist and are unique due to the continuity of x_A and x_B; the monotonicity of these functions from (9b); and the limit properties of these functions from (9c) and (9d). We next show $\sigma_A^{t+1} < \sigma_B^{t+1}$. Suppose $\sigma_A^{t+1} = \sigma_B^{t+1}$. From (9a) we have $x_A(\sigma_A^{t+1}) + x_B(\sigma_B^{t+1}) > 1$, which contradicts the definition of σ_A^{t+1} and σ_B^{t+1}. Suppose $\sigma_A^{t+1} > \sigma_B^{t+1}$. From (9b), x_A is an increasing function, so this gives $x_A(\sigma_A^{t+1}) + x_B(\sigma_B^{t+1}) > x_A(\sigma_B^{t+1}) + x_B(\sigma_B^{t+1}) > 1$, which again contradicts the definition of σ_A^{t+1} and σ_B^{t+1}. Thus $\sigma_A^{t+1} < \sigma_B^{t+1}$.

We now establish $\sigma_A^{t+1} < \sigma^t < \sigma_B^{t+1}$. Using the monotonicity of x_A and x_B in (9b), this holds iff $x_A(\sigma_A^{t+1}) < x_A(\sigma^t)$ and $1 - x_B(\sigma^t) < 1 - x_B(\sigma_B^{t+1})$. From the definitions of σ_A^{t+1}

and σ_B^{t+1}, these inequalities hold iff $1 - x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$. We will show that the latter pair of inequalities is always satisfied.

(i) Suppose there is a (trivial or non-trivial) war in period t. This implies $L_A^t/L_B^t = (\sigma^t)^{1/(1-\alpha)}$ where (L_A^t, L_B^t) is obtained from (5d). From (13), $m_A^{t+1}/m_B^{t+1} = (\sigma^t)^{1/(1-\alpha)}$. We know L_A^t/L_B^t equalizes average products at the productivity ratio σ^t so the same is true for m_A^{t+1}/m_B^{t+1}. Proposition 1 gives $1\text{-}x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$.

(ii) Suppose there is peace in period t. From Proposition 3, this implies $n_A^t/n_B^t \in LE_P$. First consider the case in which $n_A^t/n_B^t > (\sigma^t)^{1/(1-\alpha)}$ so n_A^t/n_B^t exceeds the group size ratio that equalizes average products in period t. By (12), $m_A^{t+1}/m_B^{t+1} = \sigma^t(n_A^t/n_B^t)^\alpha$. This gives $n_A^t/n_B^t > m_A^{t+1}/m_B^{t+1} > (\sigma^t)^{1/(1-\alpha)}$. Due to $n_A^t/n_B^t \in LE_P$ and the fact that $(\sigma^t)^{1/(1-\alpha)}$ is in the interior of LE_P in period t, m_A^{t+1}/m_B^{t+1} is in the interior of the set LE_P defined by σ^t in period t. Proposition 2(b) then gives $1\text{-}x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$. A parallel argument yields the same result for the case in which $n_A^t/n_B^t < (\sigma^t)^{1/(1-\alpha)}$. The only other case is $n_A^t/n_B^t = (\sigma^t)^{1/(1-\alpha)}$, which gives $m_A^{t+1}/m_B^{t+1} = (\sigma^t)^{1/(1-\alpha)}$. Again, m_A^{t+1}/m_B^{t+1} is in the interior of the set LE_P defined by σ^t and Proposition 2(b) gives $1\text{-}x_B(\sigma^t) < m_A^{t+1}/N^{t+1} < x_A(\sigma^t)$.

This concludes the proof that $\sigma_A^{t+1} < \sigma^t < \sigma_B^{t+1}$.

When $\sigma^{t+1} < \sigma_A^{t+1}$, the monotonicity of x_A gives $x_A(\sigma^{t+1}) < x_A(\sigma_A^{t+1}) \equiv m_A^{t+1}/N^{t+1}$ or $x_A(\sigma^{t+1})/[1\text{-}x_A(\sigma^{t+1})] < m_A^{t+1}/m_B^{t+1}$. When $\sigma^{t+1} > \sigma_B^{t+1}$, the monotonicity of x_B gives $1\text{-}x_B(\sigma^{t+1}) > 1\text{-} x_B(\sigma_B^{t+1}) \equiv m_A^{t+1}/N^{t+1}$ or $[1\text{-}x_B(\sigma^{t+1})]/x_B(\sigma^{t+1}) > m_A^{t+1}/m_B^{t+1}$. When $\sigma_A^{t+1} \leq \sigma^{t+1} \leq \sigma_B^{t+1}$, we have $[1\text{-}x_B(\sigma^{t+1})]/x_B(\sigma^{t+1}) \leq m_A^{t+1}/m_B^{t+1} \leq x_A(\sigma^{t+1})/[1\text{-}x_A(\sigma^{t+1})]$. Thus one of the inequalities in (b) of the proof holds iff $\sigma^{t+1} \notin [\sigma_A^{t+1}, \sigma_B^{t+1}]$ as in Proposition 4(b).

Corollary.

From Proposition 4, $\sigma_A^{t+1} < \sigma^t = \sigma^{t+1} < \sigma_B^{t+1}$ implies that there cannot be a non-trivial war in period t+1 regardless of whether there is war or peace in period t. A trivial war can be ruled out using (i) and (ii) in the proof of Proposition 4 and substituting $\sigma^t = \sigma^{t+1}$ to show that $m_A^{t+1}/m_B^{t+1} \in LE_P$ for period t+1. Proposition 3(a) then yields peace in period t+1.